SOPHUS LIE.

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BIOGRAPHY.

SOPHUS LIE.

BY GEORGE BRUCE HALSTED.

ON the eighteenth of February, 1899, the greatest mathematician in the world, Sophus Lie, died at Christiania in Norway.

He was essentially a geometer, though applying his splendid powers of space creation to questions of analysis. From Lie comes the idea that every system of geometry is characterized by its group.

In ordinary geometry a surface is a locus of points ; in Lie's *Kugel-geometrie* it is the aggregate of spheres touching this surface. By a simple correlation of this sphere-geometry with Pluecker's line-geometry, Lie reached results as unexpected as elegant. The transition from this line-geometry to this sphere-geometry was an example of contact-transformations.

Now contact-transformations find application in the theory of partial differential equations, whereby this theory is vastly clarified. Old problems were settled as sweepingly as new problems were created and solved. Again, with his *Theorie der Transformationsgruppen*, Lie changed the very face and fashion of modern mathematics.

A magnificent application of his theory of continuous groups is to the general problem of non-Euclidean geometry as formulated by Helmholtz. To this was awarded the great Lobachevski Prize. Not even this award could sufficient-

ly emphasize the epoch-making importance of Lie's work in the evolution of geometry.

Moreover, the foundations of all philosophy are involved. To know the non-Euclidean geometry involves abandonment of the position that axioms as to their concrete content are necessities of the inner intuition ; likewise abandonment of the position that axioms are derivable from experience alone.

Lie said that in the whole of modern mathematics the weightiest part is the theory of differential equations, and, true to this conviction, it has always been his aim to deepen and advance this theory.

Now it may justly be maintained that in his theory of transformation groups Lie has himself created the most important of the newer departments of mathematics.

By the introduction of his concept of continuous groups of transformations he put the isolated integration theories of former mathematicians upon a common basis.

The masterly reach of Lie's genius is illustrated by his encompassment of the fundamentally important theory of differential invariants associated with the English names Cayley, Cockle, Sylvester, Forsyth.

Thirteen years ago Sylvester announced his conception of 'Reciprocants,' a body of differential invariants not for a group, but for a mere interchange of variables. A number of Englishmen thereupon took up investigations about orthogonal, linear and projective groups, groups in whose transformations interchanges of variables occur as particular cases, and whose differential invariants are consequently classes of reciprocants, and of the analogues of reciprocants, when more variables than two are considered.

Now all these investigations were long subsequent to Lie's consideration of the groups in question as leading cases of a general conception. Thus they were merely secondary investigations !

Again the theory of complex numbers appears as a part of the great 'Theorie der Transformationsgruppen.' Indeed, this continent of 'transformations' opened up and penetrated with such giant steps by Lie represents the most remarkable advance which mathematics in all its entirety has made in this latter part of the century.

Sophus Lie it was who made prominent the importance of the notion of group, and gave the present form to the theory of continuous groups. This idea, like a brilliant dye, has now so permeated the whole fabric of mathematics that Poincaré actually finds that in Euclid 'the idea of the group was potentially preexistent,' and that he had 'some obscure instinct for it, without reaching a distinct notion of it.' Thus the last shall be first and the first last.

In personal character Lie was our ideal of a genius, approachable, outspoken, unconventional, yet at times fierce, intractable.

His work is cut short ; his influence, his fame, will broaden, will tower from day to day.

SOPHUS LIE.*

BY PROFESSOR GASTON DARBOUX.

Sophus Lie was born on the 17th of December, 1842, at Nordfjordeid (near Florö) where his father, John Herman Lie, was pastor. The studies of his childhood and youth did not reveal in him that exceptional aptitude for mathematics which is signalized so early in the lives of the great geometers: Gauss, Abel, and many others. Even on leaving the University of Christiania in 1865, he still hesitated between philology and mathematics. It was the works of Plücker on modern geometry which first made him fully conscious of his mathematical abilities and awakened within him an ardent desire to consecrate himself to mathematical research. Surmounting all difficulties and working with indomitable energy he published his first work in 1869, and we can say that from 1870 on he was in possession of the ideas which were to direct his whole career.

At this time I frequently had the pleasure of meeting and conversing with him in Paris where he had come with his friend F. Klein. A course of lectures by Sylow revealed to Lie all the importance of the theory of substitution groups; the two friends studied this theory in the great treatise of our colleague Jordan; they saw fully the essential rôle which it would be called upon to play in all the branches of mathematics to which it had then not been applied. They have both had the good fortune to contribute by their works to impressing upon mathematical studies the direction which appeared to them to be the best.

A short note of Lie "Sur une transformation géométrique," presented to our Academy in October, 1870, contains an extremely original discovery. Nothing resembles a sphere less than a straight line and yet, by using the ideas of Plücker, Lie found a singular transformation which makes a sphere correspond to a straight line, and which consequently makes possible the derivation of a theorem relative to an ensemble of spheres from every theorem relative to an aggregate of straight lines, and vice versa. It is true that if the lines are real, the corresponding spheres are imaginary. But such difficulties are not sufficient to deter geometers. In this curious method of transformation, each property relative to asymptotic lines of a surface is transformed into a property relative to lines of curvature. The name of Lie will remain attached to these concealed relations which connect the two essential and fundamental elements of geometric investigation, the straight line and sphere. He has developed them in detail in a memoir full of new ideas which appeared in 1872 in the *Mathematische Annalen*.

The works following this brilliant beginning fully confirmed all the hopes to which it gave birth. Since the year 1872 Lie has put forth a series of memoirs upon the most difficult and most advanced parts of the integral calculus. He

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commences by a profound study of the works of Jacobi on the partial differential equations of the first order and at first coöperates with Mayer in perfecting this theory in an essential point. Then, by continuing the study of this beautiful subject, he is led to construct progressively that masterful theory of continuous transformation groups which constitutes his most important work and in which, at least at the start, he was aided by no one. The detailed analysis of this vast theory would require too much space here. It is proper, however, to point out particularly two elements wholly essential to these researches: first, the use of contact transformations which throws such a vivid and unexpected light upon the most difficult and obsure parts of the theories relative to the integration of partial differential equations; second, the use of infinitesimal transformations. The introduction of these transformations is due entirely to Lie; their use, like that of Lagrange's variation, naturally greatly extends both the notion of differential and the applications of the infinitesimal calculus.

The construction of so extended a theory did not satisfy Lie's activity. In order to show its importance he has applied it to a great number of particular subjects, and each time he has had the good fortune of meeting with new and elegant properties. I find my preference in the researches which he has published since 1876 on minimal surfaces. The theory of these surfaces, the most attractive perhaps that presents itself in geometry, still awaits, and may await a long time, the complete solution of the first problem to be proposed in it, namely, the determination of a minimal surface passing through a given contour. But, in return, it has been enriched by a great number of interesting propositions due to a multitude of geometers. In 1866 Weierstrass made known a very precise and simple system of formulæ which has called forth a whole series of new studies on these surfaces. In his works Lie returns simply to the formulæ of Monge; he gives their geometric interpretation and shows how their use can lead to the most satisfactory theory of minimal surfaces. He makes known methods which permit of determining all algebraic minimal surfaces of given class and order. Finally, he studies the following problem: to determine all algebraic minimal surfaces inscribed in a given algebraical developable surface. He gives the complete solution for the case where only one of these surfaces inscribed in the developable is known.

Of great interest also are the researches which we owe to him on the surfaces of constant curvature, in the study of which he makes use of a theorem of Bianchi on geodesic lines and circles, likewise those on surfaces of translation, on the surfaces of Weingarten, on the equations of the second order having two independent variables, et cetera. I should reproach myself for forgetting, even in so rapid a résumé, the applications which Lie has made of his theory of groups to the non-Euclidean geometry and to the profound study of the axioms which lie at the basis of our geometric knowledge.

These extensive works quickly attracted to the great geometer the attention of all those who cultivate science or are interested in its progress. In 1877 a new chair of mathematics was created for him at the University of Christiania,

and the foundation of a Norwegian review enabled him to pursue his work and publish it in full. In 1886, he accepted the honor of a call to the University of Leipzig; he taught in this university with the rank of ordinary professor from 1886 to 1898. To this period of his life is to be referred the publication of his didactic works, in which he has coördinated all his researches. Six months ago he returned to his native land to assume at Christiania the chair which had been especially reserved for him by the Norwegian parliament, with the exceptional salary of ten thousand crowns. Unfortunately, excess of work had exhausted his strength and he died of cerebal anæmia at the age of fifty-six years.

Nowhere is his loss felt more keenly than in our country, where he had so many friends. True, in 1870 a misadventure befell him, whose consequences I was instrumental in averting. Surprised at Paris by the declaration of war, he took refuge at Fontainebleau. Occupied incessantly by the ideas fermenting in his brain, he would go every day into the forest, loitering in places most remote from the beaten path, taking notes and drawing figures. It took little at this time to awaken suspicion. Arrested and imprisioned at Fontainebleau, under conditions otherwise very comfortable, he called for the aid of Chasles, Bertrand, and others; I made the trip to Fontainebleau and had no trouble in convincing the procureur impérial; all the notes which had been seized and in which figured complexes, orthogonal systems, and names of geometers, bore in no way upon the national defenses. Lie was released; his high and generous spirit bore no grudge against our country. Not only did he return voluntarily to visit it but he received with great kindness French students, scholars of our École Normale who would go to Leipzig to follow his lectures. It is to the École Normale that he dedicated his great work on the theory of transformation groups. A number of our thesis at the Sorbonne have been inspired by his teaching and dedicated to him.

The admirable works of Sophus Lie enjoy the distinction, to-day quite rare, of commanding the common admiration of geometers as well as analysts. He has discovered fundamental propositions which will preserve his name from oblivion, he has created methods and theories which, for a long time to come, will exercise their fruitful influence on the development of mathematics. The land where he was born and which has known how to honor him can place with pride the name of Lie beside that of Abel, of whom he was a worthy rival and whose approaching centenary he would have been so happy in celebrating.